Extraneous Variables and the Interpretation of Regression Coefficients

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This paper addresses some difficulties concerning the interpretation of the regression coefficients in simple and multiple regression models. The root of the problem lies in the fact that the fitted multiple regression equation is the result of transforming raw data of the independent variables into residualized scores. In the standard interpretation of the partial regression coefficients, effects of the residual term have not been explicitly differentiated from those of the regressors. Alternative interpretations of the regression coefficients are proposed. The recognition of residual and residualized effects plays an important role in the evaluation of the obtained values of the regression coefficients, R^2, the overall F tests and the construct validity of the multiple regression model.

There are three types of variables in a regression model, namely, the dependent variable (Y), at least one regressor or independent variable (X_j, j = 1, …, m) and the unknown error term (ε) estimated by the residual scores ($e = Y - \hat{Y}$) which in turn represent the extraneous variables, where $\hat{Y}$ is the predicted value of Y. Typically, the regression slope coefficient in the *simple* regression model $\hat{Y} = a + bX$ is defined as, "the amount of the difference in $\hat{Y}$ associated with a one-unit difference in X" (Howell, 1997, p. 242), or "The slope of the line equals the gain in Y associated with each 1-unit gain in X" (Darlington, 1990, p.10). On the other hand, each of the slope coefficients in the *multiple* regression model $\hat{Y} = a + b_1X + \ldots + b_mX_m$ is called a *partial regression coefficient* "to make clear that it is the weight to be applied to an independent variable (*IV*) when one or more specified *IV*s are also in the equation" (Cohen & Cohen, 1983, p. 83). The coefficient b_j, j = 1, 2,…, m, is defined as, "the change in the dependent variable per unit change in the jth independent variable, *assuming all other independent variables are held constant*" (Rawlings, 1988, p. 67). Similar definitions are found in several textbooks on regression analysis. It will be argued in this paper that the above definitions of b and b_j should be used with great care to avoid misleading interpretation on the effects of X_j in predicting Y for data analysis. First, some possible implications of "holding all regressors but one constant" in the multiple regression model are explored. Next, in an attempt to understand the meanings of regression coefficients, several ways to obtain their estimates are investigated. It will be demonstrated that the independent variables can be operationally transformed into the residualized terms in the process of computing the partial regression coefficients. This leads to the realization that a regression analysis transforms the obtained data into another data set called the residualized scores while reproducing the same values for the partial regression coefficients. As a result, a simple way to determine the residualized scores in multiple regression models is developed.

Before proceeding, however, an explanation of the terms "residual scores" and "residualized scores" is in order. The *residual term* (e) represents the difference between Y and $\hat{Y}$ as a result of regressing Y against one or more independent variables (X's); denoted as $e_{Y,1}$, $e_{Y,2}$, or $e_{Y,1,2}$ for regression models involving one or two regressors (where the first subscript represents the dependent variable and the subsequent subscripts, the independent variables). A *residualized variable* is formed when the residual term (e) is used either as a regressor (E_j) or as a dependent variables yielding predicted values ($\hat{e}_{Y,j}$ and $\hat{e}_{Y,h}$). The residual scores (e) capture the portion of variability in Y, called the "uncontrolled" extraneous effect of the model, that is not accounted for by *all* independent variables (X's). On the other hand, the residualized scores of X_j, say E_j (for any j = 1, …, m), represent the residual term when X_j is regressed on all other independent variables. Thus, when Y is regressed on the jth residualized variable, the resulting regression coefficient represents only the effect of X_j since the effects of other independent variables in the original multiple regression model have been "partialled out."

Winne (1969) has studied the problems of construct validity in using multiple regression models. He indicated that regressors in such models do not represent the constructs described by the original data since the partial regression coefficients are computed for the residualized scores instead. However, he did not discuss how these residualized scores can be interpreted and analyzed. Rather, Winne (1969) recommended that, "anchor variables not of direct interest in a research study be measured and correlated with residualized variables. This supplementary analysis sheds light on changes to construct validity that must be known before interpreting multiple regression analyses" (p. 187). On the contrary, it is suggested in this paper that the effects of regressors in multiple regression models are interpreted as those of residualized scores, in the same way as one would interpret partial correlation coefficients. Then, the simple regression equations of Y on the residualized variables (called the "residualized regression equations") are studied to shed light on the interpretation of the partial regression coefficients in the conventional multiple regression equation. Moreover, the coefficient of determination associated with the multiple regression model is explained in terms of the *semi-partial coefficients of determination* obtained from the above residualized regression equations. Finally, the residual plots of the multiple regression model and those of the residualized regression equations (called the *partial regression residual plots*) are examined for model diagnostics. These steps are recommended not only for identifying the effects of "controlled" extraneous variables associated with the partial regression coefficients and recapture the same R^2 but also for obtaining test statistics (t for the slopes and overall F for the fit) that take into account the influence of the residual and residualized variables. For the sake of illustration, all numerical analyses are based on the data set in Figure 1, Panel A. In the multiple regression under consideration, Test Score (Y) is regressed on Cumulative GPA (X_1) and Study Hour (X_2).

Limitations of the Conventional Interpretations

What Happens to Partial Regression Coefficients If Only Values of One Regressor Are Changed?

The main difference in the definitions of simple and partial regression coefficients given above lies in the requirement that all but one regressors in the multiple regression model are "held constant." It is true that if values of the jth regressor $X_{j,i}$ in the multiple regression equation *for the ith subject* (i = 1, 2, …, n) is changed by a constant whereas the observed values of remaining regressors are intact then the predicted value $\hat{Y}_i$ for this particular subject is modified by an amount of b_j. However, if values of X_j in the above example are modified by a fixed constant *for all subjects* then in the resulting regression equation, only the intercept term (a) will change (i.e., values of $\hat{Y}_i$ and all slopes b_1, …, b_m remain the same for i = 1, …, n). Although only values of a single regressor have been modified, one no longer has the same regression model since the intercept term has changed. The following example serves to illustrate this point.

Based on the data set in Figure 1, Panel A, three regression models are considered, the first with the original values for Y, X_1 and X_2 and the remaining two, with the linearly transformed values of $X_3 = X_1 + 5$ and $X_4 = X_2 + 3$. As expected, the resulting regression equations have the same slopes but different intercepts:

Model 1a: $\hat{Y} = a_1 + b_1X_1 + b_2X_2$
$= 43.651 + 7.301X_1 + 2.839X_2$, $R^2 = .4105$,

Model 2: $\hat{Y} = a_2 + b_1X_3 + b_2X_2$
$= 7.145 + 7.301X_3 + 2.839X_2$, $R^2 = .4105$,

Model 3: $\hat{Y} = a_3 + b_1X_3 + b_2X_4$
$= -1.373 + 7.301X_3 + 2.839X_4$, $R^2 = .4105$.

Figure 1. Data and Test Statistics for Regression Models 1a and 1b

Panel A: Data Example

ID	Y	X_1	X_2
1	73.5	3.5	2.4
2	69.0	2.8	2.5
3	85.5	3.0	5.5
4	82.0	3.7	3.1
5	90.0	3.9	5.2
6	84.0	3.1	5.5
7	86.5	3.3	7.1
8	74.5	2.9	3.6
9	71.5	3.1	5.5
10	75.5	3.6	4.4
11	80.0	4.0	5.1
12	91.8	3.5	4.2
13	86.5	3.3	7.2
Mean	81.76	3.36	1.50
SD	7.96	0.36	1.50
	$r_{Y,1} = .354$	$r_{1,2} = .354$	$r_{Y,2} = .354$

Panel B: Common Statistics for Models 1a and 1b

IV	*SE*	t ($p < t$)
C/Intercept	17.941	2.433 (.0322)
X_1	5.082	1.437 (.1786)
X_2	1.232	2.305 (.0416)

Panel C: Goodness-of-Fit Statistics for Models 1a and 1b

Model	SSR (SST)	R^2	MSR (MSE)	F ($p < F$)
1a	338.15 (823.77)	0.41	167.08 (44.15)	3.78 (.0547)
1b	93933.73 (94419.35)	0.99	31311.24 (44.15)	709.20 (.0001)

Note. *IV* = Independent variable, *SE* = Standard error of the regression coefficient estimate, *SSR* = Regression sum of squares, *SST* = Total sum of squares, *MSR* = Regression mean squares, *MSE* = Error mean square.

(Note that $a_2 = a_1 - 5b_1$ and $a_3 = a_1 - 5b_1 - 3b_2$.) For example, given $X_1 = 3.50$ and $X_2 = 2.40$ for the first subject then $\hat{Y} = 76.02$ and $e = 2.52$ in the three models. Typically, the same interpretation applies to the partial regression coefficients in these models, say, "If X_2 is held constant, then each 1-unit increase in X_1 leads to an average increase in $\hat{Y}$ of 7.301 units." However, if the units of measurement for any independent variable has been changed, not only a new regression model with a different intercept term is needed but also the statistical significance of the intercept term may also be altered (In the three models above, $t(\alpha) = 2.433$, $p < .03$, $t(\alpha) = 0.167$, $p > .870$

and $t(\alpha)$ = -0.032, p > .975, respectively). Apparently, the standard interpretation is focussed on the case in which one wants to compute a predicted value $\hat{Y}_i$, given a certain value of $X_{j,i}$, i = 1, 2, …, n, for each subject, one at a time. However, the intercept term should be considered when the scales of measurement have been changed for several, if not all, subjects. Hence, the above statement could be modified as, "For each observation i, if $X_{2,i}$ is held constant, then a 1-unit increase in $X_{1,i}$ leads to an increase in $\hat{Y}_i$ of 7.301 units. On the other hand, if all values of X_2 are changed by the same constant c, then a 1-unit increase in $X_{1,i}$ leads to an average increase in $\hat{Y}_i$ of (43.651 - 7.301c) units."

The Same Values of Partial Regression Coefficients May Not Yield the Same Regression Models

The above three regression models have different values for the intercept term but otherwise identical with respect to the test statistics of t for regression coefficients as well as overall F and R^2 for goodness of fit (as reported for Model 1a in Figure 1, Panel C). The regression coefficients in Model 1a can be reproduced by regressing Y on C, X_1 and X_2 where C is a dummy variable of constant values, say $C = 1$,

Model 1b: $\hat{Y} = a_1C + b_1X_1 + b_2X_2$
$= 43.651C + 7.301X_1 + 2.839X_2$, $R^2 = .995$.

Although the t tests for the regression coefficients in Models 1a and 1b are identical (Figure 1, Panel B), they are substantially different with respect to goodness-of-fit statistics (Figure 1, Panel C). Model 1a yields poor fit with small R^2 and marginally significant overall F. In Model 1a, R^2 represents the ratio of sum of squares of regression (SSR) over the *corrected* total sum of square (SST_c). Since Model 1b has no intercept term, R^2 has been redefined by using the *uncorrected* total sum of square (SST_u). As a result, both its R^2 and F have increased remarkably! It can be explained that this phenomenon occurs when extraneous effects independent of the predictors have been accounted for in the regression model. Hence the significance test of the failure to control for the impact of extraneous variables under the null hypothesis can be conducted by means of the following F test with degrees of freedom (q, df_r):

$$F = \frac{R^2(\text{Model.1b}) - R^2(\text{Model.1a})}{1 - R^2(\text{Model.1b})}\left[\frac{df_r}{q}\right],$$

where df_r = the residual degree of freedom in Model 1b and q = (the difference in number of regressors in Models 1b and 1a) = 1 (Darlington, 1990, pp. 124-125; Cohen and Cohen, 1983, pp. 145-151). For the data at hand, F = (.9610 - .2652)(11) = 7.653, p < .00001.

Different Ways to Obtain Values of the Regression Coefficients

In an attempt to enhance the understanding, and thus improving the interpretations, of simple and partial regression coefficients, it is necessary to investigate several ways to obtain the same values of these coefficients for a given data set. Some of the steps presented below have been discussed elsewhere (Draper and Smith, pp. 196-201) but for a different objective, namely, the confirmation of the least-squares results by various methods rather than the difference in their interpretations.

As presented in Table 1, thirteen regression models can be computed on the basis of two predictors X_1 and X_2 (in Figure 1, Panel A). The three models g, h and k are the pivot models against which all remaining models will be compared. For identification purposes, the subscripts "g", "h" and "k" are attached to the regression coefficients when necessary. The predicted values $\hat{Y}$ and $\hat{X}_j$, j = 1, 2, in steps h, k, 4 and 5 are used as dependent variables ($\hat{Y}_{Y,1}$, $\hat{Y}_{Y,2}$) or regressors ($\hat{X}_1$, and $\hat{X}_2$) in steps 6 and 7, respectively. For the remaining models (steps 8 to 13), either the residual scores (obtained in steps h and k) or Y are regressed on the residualized scores (E_j obtained in steps 4 and 5) and X_j. The intercept terms are present in all regression models with raw data, except in steps 8 and 9 where only the residual and residualized scores are involved.

The regression models in Table 1 were computed using both raw and standardized data with identical variables. All the regression models with standardized scores must be fitted without the intercept terms (The computed values of the intercept terms would be zero had they been included). The results in Table 2 illustrate that it is the variable type, not the data metric, which determines the elements constituting the "extraneous variables."

Approach 1 (Based on Raw Data)

The *simple* regression coefficients for X_1 and X_2 are 7.775 (step h) and 2.911 (step k), respectively. Their *partial* counterparts are $b_{Y,1.2}$ b_{1g} = 7.301 and $b_{Y,2.1}$ b_{2g} = 2.839 (step g).

Approach 2 (Based on Standardized Scores)

For *simple* regression models (in steps h and k), the *simple*, or *zero-order*, correlations of Y and X_j are used instead of b_j (i.e., $r_{Y,1}$ r_h = .3545 and $r_{Y,2}$ r_k = .5476). For *multiple* regression models, each $r_{Y,j.i}$ denotes the *partial* correlation of Y and X_j, or the correlation of Y and X_j, given that X_i has already entered the model ($r_{Y,1.2}$ r_{1g} = .3329, and $r_{Y,2.1}$ r_{2g} = .5341 in step g).

Table 1. Regression Models for Comparing Regression/Correlation Coefficients and R^2.

Step	Regression Models
g	Y is regressed on X_1 and X_2
h	Y is regressed on X_1 (yielding $\hat{Y}_{Y,1}$ and $e_{Y,1}$)
k	Y is regressed on X_2 (yielding $\hat{Y}_{Y,2}$ and $e_{Y,2}$)
4	X_1 is regressed on X_2 (yielding $\hat{X}_1$ and E_1)
5	X_2 is regressed on X_1 (yielding $\hat{X}_2$ and E_2)
6	$\hat{Y}_{Y,1}$ (from step h) is regressed on $\hat{X}_1$ (from step 4)
7	$\hat{Y}_{Y,2}$ (from step k) is regressed on $\hat{X}_2$ (from step 5)
8	$e_{Y,1}$ (from step h) is regressed on E_1 (from step 4) (without the intercept term)
9	$e_{Y,2}$ (from step k) is regressed on E_2 (from step 5) (without the intercept term)
10	Y is regressed on E_1 (from step 4)
11	Y is regressed on E_2 (from step 5)
12	Y is regressed on X_2 and E_1 (from step 4)
13	Y is regressed on X_1 and E_2 (from step 5)

Approach 3 (Based on Predicted and Residualized Scores)

The same values of the slope/correlation coefficients in the simple and multiple regression models can also be obtained by fitting regression models on the basis of predicted ($\hat{Y}$) and residualized scores. In step 6, by regressing $\hat{Y}_{Y,1}$ obtained in step h on $\hat{X}_1$ in step 4, the *simple* regression/correlation coefficients in step h are recovered. Similarly, the results in step k are reproduced in step 7 by regressing $\hat{Y}_{Y,2}$ (step k) on $\hat{X}_2$ (step 5). The two *partial* regression/correlation coefficients in step g are reclaimed by fitting two simple regression models in terms of residualized scores ($\hat{e}_{Y,j}$ and E_j) in steps 8 and 9, respectively.

Identifying the Extraneous Variables in the Multiple Regression Model

What Are the Residualized Scores for X_j?

A much simpler procedure to obtain the residualized scores for any regressor and show that its effect can be measured by the corresponding partial regression coefficient is described below.

Step (i). Fit X_j on the remaining regressors:

$$\hat{X}_j = a + b_1X_1 + \ldots + b_{j-1}X_{j-1} + b_{j+1}X_{j+1}.$$

For Model 1a, this is realized by obtaining the regression equations in steps 4 (for X_1) and 5 (for X_2) in Table 2.

Step (ii). Obtain the residualized scores for X_j:

$$E_j = X_j - \hat{X}_j \text{ for } j = 1, 2, \ldots, m.$$

Thus, in the example, the residual terms obtained by fitting the regression equations in steps 4 and 5 (Table 2) yield the residualized scores for X_1 and X_2, respectively.

Step (iii). Reproduce the partial regression coefficient for X_j, by fitting the regression equations in steps 8 and 9 (or steps 10 and 11, Table 2). Alternatively, they can be computed as:

$$b_j = \mathrm{Cov}(Y, E_j)/S^2(E_j) = r(Y,E_j)\{S(Y)/S(E_j)\},$$

where E_j = the jth residualized variable, $\mathrm{Cov}(Y, E_j)$ = covariance of Y and E_j, $S^2(E_j)$ = sample variance of E_j, $S(Y)$ = sample standard deviation of Y, and $r(Y,E_j)$ = the zero-order correlation of Y and E_j. For the example, $S(Y) = 7.9603$, $S(E_1) = .3626$, $S(E_2) = 1.496$, $r(Y,E_1) = .3325$ and $r(Y,E_2) = .5337$. Therefore, the partial regression coefficients for X_1 and X_2 are:

$b_1 = r(Y,E_1)\{S(Y)/S(E_1)\} = (.33)(7.96)/.36 = 7.30$, and $b_2 = r(Y,E_2)/\{S(Y)/S(E_2)\}=(.53)(7.96)/1.49 = 2.84$, respectively.

Understanding the Simple Regression Correlation Coefficient

So far, the regression and correlation coefficients in a simple regression model play the same roles. The same values of simple regression/correlation coefficients are reproduced in steps h and 6 (Table 2) because the predicted values of $\hat{Y}$ and $\hat{Y}_{Y,1}$ in these equations are determined by X_1 independently of X_2. Analogously, the regression/correlation coefficients in steps k and 7 are identical since the relevant predicted values of $\hat{Y}$ and $\hat{Y}_{Y,2}$ are determined by X_2 and free of X_1. Since the different values of regression and correlation coefficients are simply due to data metrics, their meanings should be interpreted similarly. The simple correlation coefficient has been defined as "a measure of the degree of closeness of the linear relationship between two variables" (Snedecor and Cochran, 1967, p. 173). This statement remains meaningful in the context of simple regression models with either raw or standardized scores. A linear relationship is one in which the variation in Y, produced by a specified change in X, is constant. The "linear relationship" between X and $\hat{Y}$ in the simple regression model with the intercept has two components, constant (determined by the intercept) and linearly changeable (accounted for by the slope). Therefore, the slope regression coefficient in a simple regression model can be interpreted as, "*In raw data metric, the slope b represents the relative weight of X to account for the linear variability in $\hat{Y}$ that is free of the unknown extraneous effects represented by the residual $e = Y - \hat{Y}$.*"

In other words, bX represents the linear trend of, or portion of the linear variation in, the values of $\hat{Y}$ that is not attributed to unknown extraneous effects.

Table 2. Results and Test Statistics for the Regression Models in Table 1.

Step	Regression Model (Raw Data)	Regression Model (Standardized)	SSR (SST)	R^2	MSR	MSE	F ($p<F$)
g	$\hat{Y} = a_g + b_{1g}X_1 + b_{2g}X_2$ $= 43.652 + 7.301X_1 + 2.839X_2$	$\hat{Y} = r_{1g}X_1 + r_{2g}X_2$ $= .3329X_1 + .5341X_2$	338.15 (823.77)	.4105	169.07	44.15	3.83 (.05)
h	$\hat{Y} = a_h + b_hX_1$ $= 55.606 + 7.775X_1$	$\hat{Y} = r_hX_1 = .3545X_1$	103.50 (823.77)	.1256	103.50	60.06	1.72 (.00)
k	$\hat{Y} = a_k + b_kX_2$ $= 67.874 + 2.911X_2$	$\hat{Y} = r_kX_2 = .5476X_2$	247.03 (823.77)	.2999	247.03	48.06	5.14 (.04)
4	$\hat{X}_1 = a_1 + b_{1,2}X_2$ $= 3.317 + .010X_2$	$\hat{X}_2 = r_{2,1}X_1$ $= .0404X_1$	0 (1.71)	.0016	0	0.14	0.02 (.89)
5	$\hat{X}_2 = a_2 + b_{2,1}X_1$ $= 4.210 + .167X_1$	$\hat{X}_1 = r_{1,2}X_2$ $= .0404X_2$	0.05 (29.15)	.0016	0.04	2.42	0.02 (.89)
6	$\hat{Y}_{Y,1} = a_h + b_h\hat{X}_1$ $= 55.606 + 7.775\hat{X}_1$	$\hat{Y}_{Y,1} = r_h\hat{X}_1$ $= .3545\hat{X}_1$	0.17 (103.50)	.0016	0.17	8.61	0.02 (.89)
7	$\hat{Y}_{Y,2} = a_k + b_k\hat{X}_2$ $= 67.874 + 2.911\hat{X}_2$	$\hat{Y}_{Y,2} = r_k\hat{X}_2$ $= .5476\hat{X}_2$	0.40 (247.03)	.0016	0.40	20.55	0.02 (.89)
8	$\hat{e}_{Y,1} = b_{1g}E_1$ $= 7.301E_1$	$\hat{e}_{Y,1} = r_{1g}E_1$ $= .3329E_1$	234.65 (720.27)	.1580	234.65	37.36	6.28 (.03)
9	$\hat{e}_{Y,2} = b_{2g}E_2$ $= 2.839E_2$	$\hat{e}_{Y,2} = r_{2g}E_2$ $= .5341E_2$	91.12 (576.74)	.3258	91.12	37.36	2.44 (.14)
10	$\hat{Y} = \overline{Y} + b_{1g}E_1$ $= 81.764 + 7.301E_1$	$\hat{Y} = r_{1g}E_1$ $= .3329E_1$	91.12 (823.77)	.1106	91.12	61.05	1.49 (.25)
11	$\hat{Y} = \overline{Y} + b_{2g}E_2$ $= 81.764 + 2.839E_2$	$\hat{Y} = r_{2g}E_2$ $= .5341E_2$	234.65 (823.77)	.2840	234.65	49.09	4.78 (.05)
12	$\hat{Y} = a_k + b_kX_2 + b_{1g}E_1$ $= 67.874 + 2.911X_2 + 7.301E_1$	$\hat{Y} = r_kX_2 + r_{1g}E_1$ $= .5476X_2 + .3329E_1$	338.15 (823.77)	.4105	169.07	44.15	3.83 (.05)
13	$\hat{Y} = a_h + b_hX_1 + b_{2g}E_2$ $= 55.606 + 7.775X_1 + 2.839E_2$	$\hat{Y} = r_hX_1 + r_{2g}E_2$ $= .3545X_1 + .5341E_2$	338.15 (823.77)	.4105	169.07	44.15	3.83 (.05)

Note. For variables with double subscripts, the first subscript refers to the dependent variable and the second subscript denotes the regressor.

Moreover, $a + bX$ constitutes the value of $\hat{Y}$ with the maximum value of R^2 if it can be assumed that the influence of the unknown extraneous effects is equally distributed to all members of the sample. This assumption can be checked by running the regression model without the intercept that contains X and a dummy variable C of fixed values. As a result, the same value for b in the original simple regression equation is reproduced in the multiple regression model without the intercept. For the regression models in steps h and k, by letting $C = 1$ for all subjects, say, we get

$$\hat{Y} = a_hC + b_hX_1 = 55.606C + 7.775X_1,$$
$$R^2 = .9954, \quad \text{(revised step h)}$$
$$\hat{Y} = a_kC + b_kX_2 = 67.874C + 2.911X_1,$$
$$R^2 = .9939, \quad \text{(revised step k)}$$

The values of R^2 have been increased dramatically (as compared to those reported for steps h and k in Table 2) to reflect the fact that extraneous effects have been (artificially or statistically) controlled.

Understanding the Partial Regression Correlation Coefficient

The residual term in step h is regressed on the residual term in step 4 to produce the residualized scores $\hat{e}_{Y,1}$ in step 8, representing the portion of variation in Y that is free of X_2. Analogously, the residual terms in steps k and 5 are used to yield the residualized scores $\hat{e}_{Y,2}$ in step 9 representing the part of variation in Y that is not influenced by X_1. Therefore, b_{1g} (in steps g and 8) denotes the *relative weight* of X_1 in the raw data metric (or r_{1g}, the *simple correlation* between Y and X_1 in terms of standardized scores) *that is free of* X_2. Analogously, b_{2g} (in steps g and 9) signifies the *relative weight* of X_2 in raw data metric (or r_{2g}, the *simple correlation* between Y and X_2 based on standardized scores) *that is free of* X_1. Since the partial regression coefficients in step g can be reclaimed as two simple regression coefficients in steps 8 and 9, the simple and partial regression coefficients should be logically defined and explained similarly. This is the approach adopted in the following discussion.

The partial correlation coefficient, say $r_{1,2.3}$, is commonly defined as, "the correlation between variables 1 and 2 in a cross section in individuals *all having the same value* of variable 3" (Snedecor and Cochran, 1967, p. 400). In the regression context, the partial correlation $r_{Y,1\ .\ 2,\ ...,\ m}$, say, represents the portion of the correlation of Y and X_1 which has no dependence on values of the variables X_2, ..., X_m and extraneous effects. In the same vein of logic, the partial regression coefficients for X_j in a multiple regression model can be interpreted as, "*In raw data metric, the slope b_j represents the relative weight of X_j to account for the linear variability in $\hat{Y}$ that is free of the effects due to other regressors in the model and the unknown extraneous effects*."

The effects due to other regressors are estimated by the residualized scores $\hat{e}_{Y,j}$ (steps 8 and 9) whereas the unknown extraneous effects are estimated by the residual $\hat{Y}$ - Y (step g). The meaning of this interpretation is further explained by the two multiple regression equations in steps 12 and 13. The partial regression coefficient b_k in step 12 represents the simple regression coefficient (or relative weight) of X_2 whereas b_{1g}, the partial regression coefficient in step g, is actually transformed into an extraneous effect, being the slope of a residualized variable (E_1). A similar interpretation applies to b_h and b_{2g} in step 13. The transformation of regressors into residual and residualized variables in the multiple regression models is not expected to influence the test statistics. Indeed, as shown in Table 2, R^2, the sum of squares, mean squares, and F of the three models g, 12 and 13 are identical.

As shown above, had the extraneous effects been "controlled" by a dummy variable, say C = 1, the regression model in step g can be reproduced but with a much greater value of R^2.

Implications of Taking Extraneous Variables into Consideration

There are at least three pertinent outcomes rendered by the recognition of extraneous effects in the regression model: (i) an understanding of the limitations in the construct validity of multiple regression models, (ii) a proper decomposition for the coefficient of determination (R^2), and (iii) an improvement in the evaluation of estimates of the regression/correlation coefficients and the overall F tests.

Construct Validity in Multiple Regression Analysis

The extent to which the regressors can be used to meaningfully explain and accurately predict values of the dependent variable represents the construct validity of the regression model. The analysis so far indicates that, although the regressors (X_1, ..., X_m) are used in the multiple regressions, the regression slopes and R^2 measure the contributions of the residualized scores (E_1, ..., E_m) or a mixture between regressors and residualized scores unless X_1, ..., X_m are uncorrelated. Hence, the construct validity in multiple regression analysis may be low. The following illustration is adapted from Winne (1989), given the results in Table 2. From the three basic regression models (in steps g, h and k), how do the relationships among Y, X_1 and X_2 be explained? One may be tempted to arrive at the following conclusions:

(i) If the entry order was X_1 and X_2 then X_1 accounted for 12.56% of the variability in Y and X_2 accounted for an additional 28.49% of the variability in Y (since R^2 = .4105 in step g, R^2 = .1256 in step h and .4105 - .1256 = .2849). On the other hand, if the entry order was X_2 and X_1 then X_2 accounted for 29.99% of the variability in Y and X_1 accounted for the remaining 11.06% of the variance in Y.

(ii) When all variables are transformed to standardized scores, an increment of one standard deviation in X_1 is associated with a 33.29% increase in Y. Similarly, an increment of one standard deviation in X_2 yields an increase in Y by 53.41 percent.

Although intuitively meaningful, both of these statements are wrong with respect to the revised interpretations of partial regression/correlation coefficients! In the first statement (i), for the (X_1, X_2)-entry order, X_2 did not account for the additional 28.49% of the variability in Y but the residualized scores E_2 did. The statement is correct if X_2 is replaced by E_2. This can be seen by following the series of equations in steps h, 11 and 13 (either raw data or standardized scores). The last model (step 13) contains the same values for the slopes and the sum of R^2's reported for the combination of models h and 11. Similar arguments apply to the (X_2, X_1)-entry order in the second part of statement (i) above based on the results for steps k, 10 and 12. Statement (ii) is wrong since X_1 and X_2 are correlated. The statement is correct by either of the following modifications. First, the percentages are changed to 35.45% and 54.76% for X_1 and X_2, respectively (see steps h and k). The pairs (33.29%, 53.41%) and (35.45%, 54.76%) are quite close to each other since the correlation between X_1 and X_2 is quite small ($r_{1,2}$ = .04). Greater difference is expected for larger $r_{1,2}$. Alternatively, X_1 and X_2 are replaced by E_1 and E_2, respectively (see steps 10 and 11).

The mistakes made in statements (i) and (ii) presage a serious error that materializes when one attempts to assess the statistical significance of the slopes and determine the proportional contributions of the regressors to variations in Y in multiple regression models. For these purposes, the results of Table 2 should be obtained and examined in conducting the statistical evaluation of the standard multiple regression model. In particular, the regression models in steps 10 and 11 in terms of residualized scores should be used for studying the statistical inference of the partial regression/correlation coefficients. We return to this point later. Meanwhile, the following discussion serves to illustrate how to analyze the multiple regression model taking into consideration the effects of residualized scores and extraneous factors.

Decomposition of the Coefficient of Determination

The decomposition of R^2 for the multiple regression model is given by Engelhart (1936) as, say for m = 3,

$$R^2 = \beta_{Y,1}{}^2 + \beta_{Y,2}{}^2 + \beta_{Y,3}{}^2 + 2\beta_{Y,1}\beta_{Y,2}r_{1,2} + 2\beta_{Y,1}\beta_{Y,3}r_{1,3} + 2\beta_{Y,2}\beta_{Y,3}r_{2,3}, \quad (1)$$

where $\_{Y,j}$ = the standardized partial regression coefficient of X_j and $r_{j,h}$ = the zero-order correlation of X_j and X_h. From this equation, it was argued that the total variance in Y is reproduced by the direct variance (indicated by the betas squared) and shared variance (denoted by twice the sum of the correlational cross products) of the regressors. Engelhart (1936) argued that the shared variance is divided among each of the regressors in the same proportions as the direct variance. Chase (1960) modified this equation to be

$$\begin{aligned} R^2 = &(\beta_{Y,1}{}^2 + \beta_{Y,1}\beta_{Y,2}r_{1,2} + \beta_{Y,1}\beta_{Y,3}r_{1,3}) \\ &+ (\beta_{Y,2}{}^2 + \beta_{Y,1}\beta_{Y,2}r_{1,2} + \beta_{Y,2}\beta_{Y,3}r_{2,3}) \\ &+ (\beta_{Y,3}{}^2 + \beta_{Y,1}\beta_{Y,3}r_{1,3} + \beta_{Y,2}\beta_{Y,3}r_{2,3}). \quad (2)\end{aligned}$$

so that "the total direct and shared variance in the criterion associated with the ith independent variable is given by the square of the beta for the ith variable, plus half of all the covariance terms in formula (1) which include the beta for the ith variable" (p. 266). The decomposition of R^2 for step g in Table 2 yields:

Direct effect of X_1: $\beta_{Y,1}{}^2 = (.3329)^2 = .11082$,
Direct effect of X_2: $\beta_{Y,2}{}^2 = (.5341)^2 = .28526$,
Shared effect of X_1 and X_2:
$[\beta_{Y,1}\beta_{Y,2}r_{1,2} = (.3329)(.5341)(.04044) = .00719]$,
Total effect of X_1: .11082 + .00719 = .11802,
Total effect of X_2: .28526 + .00719 = .29245,
The multiple coefficient of determination:
$R^2 = .11802 + .29245 = .41047$.

Although this decomposition reproduces the multiple coefficient of determination (R^2), it is not useful in determining the contribution of the residualized variables since the coefficients of determination in steps 10 and 11 (Table 2) are not equal to the total effects of X_1 and X_2, assumed in (2) as the components of R^2. Moreover, the decomposition (2) "has none of the most important properties that a "contribution to variance" has when variables are uncorrelated" (Darlington, 1968, p. 170). A more appropriate partition of R^2 is based on the semi-partial coefficients of determination. The general form of the semi-partial coefficient of determination for the jth residualized variable ($R^2_{Y,j.j*}$) is $R^2_{Y,j.j*} = R^2 - r^2_{Y,j}$, for j ≠ j* = 1, …, m, where R^2 = the (multiple) coefficient of determination of the full model and r_{Yj} = the zero-order correlation of Y and X_j. The semi-partial coefficients of determination for X_1 and X_2 in the example are $R^2_{Y,1.2} = R^2 - r^2_{Y,1} = .4105 - (.3545)^2 = .2849$, and $R^2_{Y,2.1} = R^2 - r^2_{Y,2} = .4105 - (.5476)^2 = .1106$, respectively. As a result, the coefficient of determination in the multiple regression model can be expressed as $R^2 = \{R^2_{Y,1.2} + R^2_{Y,2.1} + r^2_{Y,1} + r^2_{Y,2}\}/2$.

Effects of Extraneous Variables on Statistical Inference

In analyzing the goodness of fit of the multiple regression model, the researcher would get a clearer understanding of the role played by partial regression coefficients by fitting the conventional (step g) and residualized versions (steps 10 and 11). The model in step g has the advantage that the regressors are expressed in terms of the original unit of measurement. Hence, with a reasonable R^2, it can be used for predicting Y. However, in assessing the contributions of the regressors to variations in Y, the regression coefficients of the residualized scores in steps 10 and 11 are more meaningful and should be used.

For the multiple regression model in step g, the slope of X_2 is statistically significant at $\alpha = .05$ [$t(b_2) = 2.305$, $p < .05$] whereas that of X_1 is not [$t(b_1) = 1.437$, $p > .18$]. However, the significance of X_2 may be misleading in light of the overall F statistic ($p > .05$, Table 2). On the other hand, the regression models of Y using the residualized variables in steps 10 and 11 facilitate the evaluation of the statistical inference on the regressors in the multiple regression model (step g). Evidently, both E_1 and E_2 are not statistically significant ($p > .25$ and .05, respectively in Table 2). Whereas the multi-dimensional graph of Y against X's that also contains the regression line for the multiple model in step g is hard to draw, the regression lines of the residualized variables can be easily depicted since the simple models 10 and 11 involve only single regressors (E_1 or E_2) and their intercept term is equal to the sample mean of Y. The plot of the regression line for step 10, say, is the same as the plot of Y on X_1 at given values of X_2 (as illustrated by Mullet, 1972) but with much less effort.

Conclusions

It is suggested that the partial regression coefficient b_j represent the effect of the jth residualized variable which is computed as the difference between X_j and its predicted values obtained by regressing X_j on all other independent variables in the multiple regression model. The revised interpretations of the regression coefficients are based not only on the mathematical properties of the regression equation but also on the sources of the values reported by such an equation. The proposed interpretations of simple and partial regression coefficients reflect the same meanings conveyed by their corresponding correlation coefficients. The consideration of residualized effects in regression analysis leads to explanations that are more uniform in terminologies for both simple and partial regression coefficients. Moreover, it enables a recognition of low construct validity in regression modelling and sheds light on how to analyze the test statistics in fitting regression models. In the simple regression model, due to the lack of residualized variables, the simple regression coefficient for X_j measures its effect in predicting Y without recognizing extraneous variables. On the other hand, the partial regression coefficient for X_j measures its contribution in predicting Y when the extraneous effects to X_j generated by *all other regressors* have been explicitly accounted for. In all regression models, the remaining effects of the extraneous variables are represented by the residual term (e).

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