Seemingly Unrelated Regression (SUR) Models as a Solution to Path Analytic Models with Correlated Errors

T. Mark Beasley
University of Alabama at Birmingham

Multivariate regression requires the design matrix for each of *p* dependent variables to be the same in form. Zellner (1962) formulated Seemingly Unrelated Regression (SUR) models as *p* correlated regression equations. SUR models allow each of the *p* dependent variables to have a different design matrix with some of the predictor variables being the same. Of particular relevance to path analysis, SUR models allow for a variable to be both in the **Y** and **X** matrices. SUR models are a flexible analytic strategy and are underutilized in educational research.

Standard multivariate regression requires that each of *p* dependent variables has exactly the same design matrix such that:

$$\mathbf{Y}_{(N\text{x}p)} = \mathbf{X}_{(N\text{x}k)}\,\mathbf{B}_{(k\text{x}p)} + \boldsymbol{\varepsilon}_{(N\text{x}p)}\,, \qquad (1)$$

where **Y** is a matrix of *p* dependent variables, **X** is a *k*-dimensional design matrix, and $\boldsymbol{\varepsilon}$ is an error matrix, which is assumed to be distributed as $\mathcal{N}_{(N\text{x}p)}(\mathbf{0},\Sigma\otimes\mathbf{I}_N)$. Multivariate regression theory using ordinary least squares (OLS) assumes that all of the **B** coefficients in the model are unknown and to be estimated from the data as:

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{Y})\,. \qquad (2)$$

Multivariate Regression and Multiple Univariate Regression

Multivariate regression is not used often in behavioral research. One reason is that the matrix algebra underlying parameter estimation (2) is a column solution. Thus, whether one uses multivariate regression or *p* separate univariate regression, the regression coefficients will be the same. The differences between univariate and multivariate regression are the types of hypotheses that can be tested and the standard errors for these secondary parameters. Suppose one were to regress $p = 3$ dependent variables (y) on to $k = 2$ predictor variables (x). The omnibus null hypothesis would be that the regression coefficients for both x_1 and x_2 on all three y variables equals zero, a multivariate test with 6 degrees-of-freedom (*df*). Another hypothesis of potential interest would that x_2 has no unique relationship to any of the three y variables after controlling for x_1, a multivariate test with $df = 3$. It is important to note that if one were to construct a "multivariate" test that reduced down to only one of the *y* variables, then the results will be the same as the univariate regression, which is another reason multivariate regression is not popular.

Zellner (1962) formulated the Seemingly Unrelated Regression (SUR) model as *p* correlated regression equations. The *p* regression equations are "seemingly unrelated" because taken separately the error terms would follow standard linear OLS model form. Calculating *p* separate standard OLS solutions ignores any correlation among the errors across equations; however, because the dependent variables are correlated and the design matrices may contain some of the same variables there may be "contemporaneous" correlation among the errors across the *p* equations. Thus, SUR models are often applied when there may be several equations, which appear to be unrelated; however, they may be related by the fact that: 1) some coefficients are assumed to be the same or zero; 2) the disturbances are correlated across equations; and/or 3) a subset of right hand side variables are the same. This third condition is of particular interest because it allows each of the *p* dependent variables to have a different design matrix with some of the predictor variables being the same. SUR models allow for a variable to be both in the **Y** and **X** matrices, which has particular relevance to path analysis. SUR models are an underused multivariate technique. Using SUR models to solve path analytic models will be explicated.

SUR Model

The SUR model is a generalization of multivariate regression using a vectorized parameter model. The **Y** matrix is vectorized by vertical concatenation, **y**v. The design matrix, **D**, is formed as a block diagonal with the j^{th} design matrix, $\mathbf{X}j$, on the j^{th} diagonal block of the matrix. The model is then expressed as:

$$\text{E}[\mathbf{Y}_{(N\text{x}p)}] = \{\mathbf{X}\mathbf{1}_{(N\text{x}m_1)}\,\beta\mathbf{1}_{(m_1\text{x}1)}\,,\ \mathbf{X}\mathbf{2}_{(N\text{x}m_2)}\,\beta\mathbf{2}_{(m_2\text{x}1)}\,,\ \mathbf{X}\boldsymbol{j}_{(N\text{x}m_j)}\,\beta\boldsymbol{j}_{(m_j\text{x}1)}\,,\ \mathbf{X}\boldsymbol{p}_{(N\text{x}m_p)}\,\beta\boldsymbol{p}_{(m_p\text{x}1)}\,\}\ ; \qquad (3)$$

where mj is the number of parameters estimated (columns) by the j^{th} design matrix, $\mathbf{X}_j$.

To illustrate in matrix notation, the SUR model is laid out as:

$$\underset{(Np\text{x}1)}{E(\mathbf{y}_v)} = \underset{(Np\text{x}1)}{\overset{E(\mathbf{y}_v)}{\begin{bmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \cdots \\ \hat{\mathbf{y}}_j \\ \cdots \\ \hat{\mathbf{y}}_p \end{bmatrix}}} \begin{matrix} (N\text{x}1) \\ (N\text{x}1) \\ \\ (N\text{x}1) \\ \\ (N\text{x}1) \end{matrix} = \underset{(Np\text{x}M)}{\overset{\mathbf{D}}{\begin{bmatrix} \underset{(N\text{x}m_1)}{\mathbf{X}_1} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ & \underset{(N\text{x}m_2)}{\mathbf{X}_2} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ & & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \underset{(N\text{x}m_j)}{\mathbf{X}_j} & \cdots & \mathbf{0} \\ & (sym) & & & & \underset{(N\text{x}m_p)}{\mathbf{X}_p} \end{bmatrix}}} \underset{(M\text{x}1)}{\overset{\mathrm{B}}{\begin{bmatrix} \beta_1(m_1\text{x}1) \\ \beta_2(m_2\text{x}1) \\ \\ \beta_j(m_j\text{x}1) \\ \\ \beta_p(m_p\text{x}1) \end{bmatrix}}} ; \tag{4}$$

where M is the total number of parameters estimated over the p models, $M=\sum_{j=1}^{p} m_j$.

Estimators for the SUR Model

One approach to solving the parameter estimates is:

$$\hat{\mathrm{B}} = [\underset{(M\text{x}Np)}{\mathbf{D}'} \underset{(Np\text{x}Np)}{\mathbf{Q}^{-1}} \underset{(Np\text{x}M)}{\mathbf{D}}]^{-1} [\underset{(M\text{x}Np)}{\mathbf{D}'} \underset{(Np\text{x}Np)}{\mathbf{Q}^{-1}} \underset{(Np\text{x}1)}{\mathbf{y}_v}] \tag{5}$$

$\mathbf{Q}$ is weight matrix based on the residual covariance matrix of the $\mathbf{Y}$ variables and is formed as:

$$\underset{(Np\text{x}Np)}{\mathbf{Q}} = \underset{(p\text{x}p)}{\hat{\Sigma}} \otimes \mathbf{I}_N \,. \tag{6}$$

To elucidate, the residual covariance matrix could be computed by regressing each of the p dependent variables on to its design matrix and obtaining the residuals. The j^{th} diagonal element of $\hat{\Sigma}$ is computed by calculating the Sum of Squares for the j^{th} residual. The ij^{th} off-diagonal element is computed by taking the cross-product of the i^{th} and j^{th} residuals. These values are then divided by an estimate for the degrees-of-freedom for each element. Using matrix notation, the ij^{th} element of $\hat{\Sigma}$ is calculated as:

$$\hat{\sigma}_{ij} = \frac{1}{(N - df^*)} \mathbf{y}'_i [\mathbf{I}_N - \mathbf{H}_i][\mathbf{I}_N - \mathbf{H}_j]\, \mathbf{y}_j ; \tag{7}$$

where $\mathbf{H}j = \mathbf{X}_j(\mathbf{X}'_j\ \mathbf{X}_j)^{-1}\mathbf{X}'_j$ is the hat matrix for the j^{th} design matrix. Although there are several approaches for defining the degrees of freedom, the most common approach is to define *df** as the average of the numerator degrees-of-freedom (*df*) for the i^{th} and j^{th} models. Thus, this SUR estimator, sometimes referred to as Zellner's two-stage Aitken estimator, is an application of generalized least squares (GLS). In fact, because the residual covariance matrix is unknown and must be estimated from the data, this application is often called feasible generalized least squares (FGLS; see Timm, 2002). It should be noted that if $\mathbf{Q}^{-1}$ is removed from equation (5), or is defined as an identity matrix ($\mathbf{Q}^{-1} \equiv \mathbf{I}$), then the results will be the same as p separate univariate regression models. To develop robust standard errors or more precise estimates of $\boldsymbol{B}$, Zellner (1962) also proposed iterating the FGLS solution (IFGLS), which has the same asymptotic properties as the FGLS (Kmenta & Gilbert, 1968). To obtain maximum likelihood (ML) estimators of $\boldsymbol{B}$ and Σ, Kmenta and Gilbert (1968) employed an iterative procedure to solve the likelihood equation:

$$L(B,\Sigma \mid \mathbf{y}) = (2\pi)^{-Np/2}\, |\Sigma|^{-N/2}\, e^{\mathrm{tr}[(\Sigma^{-1}\otimes\mathbf{I})(\mathbf{y}-\mathbf{D}B)(\mathbf{y}-\mathbf{D}B)']} \tag{8}$$

Park (1993) showed that the ML and IFGLS estimators are mathematically equivalent. Kmenta and Gilbert (1968) found that the ML (IFGLS) and FGLS estimators gave similar results; however, FGLS is favored in small samples. Because the FGLS estimator is always unbiased and requires the least computation burden, it is recommended in most applications of the SUR model with small samples.

SUR Model Approach to Path Analysis

To demonstrate how a SUR model can be used to solve a path analysis problem, suppose the path model in Figure 1. The "terminal" endogenous variable is y_1, which is directly influenced by y_2, y_3, and x_2. One exogenous variable, x_2 also has indirect effects on y_1 through y_2 and y_3. The exogenous variable, x_1, has an indirect effect on y_1 though y_2. The exogenous variable, x_3, has an indirect effect on y_1 though y_3. The path diagram also models correlation among the errors of the endogenous variables. Assuming standardized variables so that all intercepts will be zero, the correctly specified regression models would be:

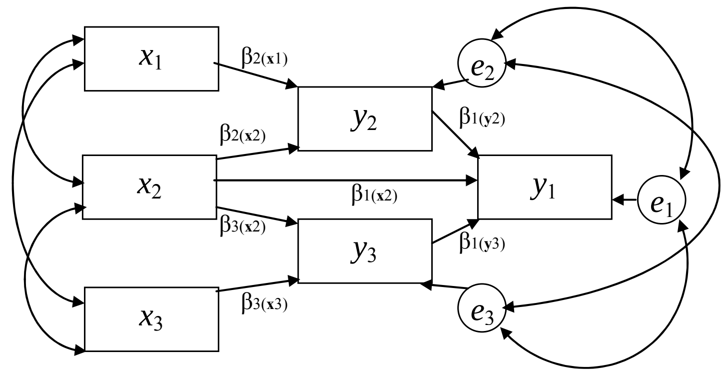

Figure 1. Hypothetical Path Model

$$\begin{aligned}
\hat{\mathbf{y}}_1 &= \beta_{1(y2)}\,\mathbf{y}_2 + \beta_{1(y3)}\,\mathbf{y}_3 + \mathbf{0}\,\mathbf{X}_1 + \beta_{1(X2)}\,\mathbf{X}_2 + \mathbf{0}\,\mathbf{X}_3 \\
\hat{\mathbf{y}}_2 &= \beta_{2(X1)}\,\mathbf{X}_1 + \beta_{2(X2)}\,\mathbf{X}_2 + \mathbf{0}\,\mathbf{X}_3 \\
\hat{\mathbf{y}}_3 &= \mathbf{0}\,\mathbf{X}_1 + \beta_{3(X2)}\,\mathbf{X}_2 + \beta_{3(X3)}\,\mathbf{X}_3
\end{aligned} \tag{9}$$

The first subscript refers to the dependent variable (y) and the second subscript in parentheses refers to the predictor variable. For example, $\beta_{1(y3)}$ refers to the regression coefficient (path) of y_3 to y_1. Because the dependent variables and their error terms are correlated and the design matrices contain some of the same variables there is "contemporaneous" correlation among the errors across the p equations. However, the standard OLS solutions will ignore any correlation among the errors across these three equations.

Appendix B shows SAS/IML code for generating data for the path model in Figure 1. The sample size was set at N = 5000 so that asymptotical properties could be observed. The correlations among the exogenous **X** variables were set at $r_{x12} = 0.30$, $r_{x13} = 0.25$., and $r_{x23} = 0.15$. Table 1 displays the other preset coefficients.

Solving Parameter Estimates for SUR Models

The correctly specified SUR model for this path analytic problem would be laid out as such:

$$\underset{(3N\text{x}1)}{E(\mathbf{y}_v)} = \underset{(3N\text{x}1)}{\overset{E(\mathbf{y}_v)}{\begin{bmatrix} \hat{\mathbf{y}}_{11} \\ \hat{\mathbf{y}}_{12} \\ \cdots \\ \hat{\mathbf{y}}_{1N} \\ \hat{\mathbf{y}}_{21} \\ \hat{\mathbf{y}}_{22} \\ \cdots \\ \hat{\mathbf{y}}_{2N} \\ \hat{\mathbf{y}}_{31} \\ \hat{\mathbf{y}}_{32} \\ \cdots \\ \hat{\mathbf{y}}_{3N} \end{bmatrix}}} = \underset{(3N\text{x}7)}{\overset{\mathbf{D}}{\begin{bmatrix} \mathbf{y}_{21} & \mathbf{y}_{21} & \mathbf{x}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{y}_{22} & \mathbf{y}_{22} & \mathbf{x}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & & & & \\ \mathbf{y}_{2N} & \mathbf{y}_{21} & \mathbf{x}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & (N\text{x}3) & & \mathbf{x}_{11} & \mathbf{x}_{21} & \mathbf{0} & \mathbf{0} \\ & & & \mathbf{x}_{12} & \mathbf{x}_{22} & \mathbf{0} & \mathbf{0} \\ & & & \cdots & \cdots & & \\ & & & \mathbf{x}_{1N} & \mathbf{x}_{2N} & \mathbf{0} & \mathbf{0} \\ & (sym) & & (N\text{x}2) & & \mathbf{x}_{21} & \mathbf{x}_{31} \\ & & & & & \mathbf{x}_{22} & \mathbf{x}_{32} \\ & & & & & \cdots & \cdots \\ & & & & (N\text{x}2) & \mathbf{x}_{2N} & \mathbf{x}_{3N} \end{bmatrix}}} \underset{(7\text{x}1)}{\overset{\mathbf{B}}{\begin{bmatrix} \beta_{1(y2)} \\ \beta_{1(y3)} \\ \beta_{1(X2)} \\ \beta_{2(X1)} \\ \beta_{2(X2)} \\ \beta_{3(X2)} \\ \beta_{3(X3)} \end{bmatrix}}} \tag{10}$$

Setting this path analysis model up as a SUR model allows for the simultaneous solution of the coefficients in closed form and will produce estimates of the standard errors that take the contemporaneous correlations into account.

Appendix C shows code for the SYSLIN, CALIS, and MIXED modules of SAS. In the PROC SYSLIN code the FIML option produces the Full Information Maximum Likelihood estimates. Other estimation methods include the SUR option, which produces the FGLS estimates, and the ITSUR (ITterative SUR) option, which produces the IFGLS estimates.

Table 1. Parameter Estimates from SAS PROC SYSLIN, CALIS, and MIXED.

Coefficients for:	y_1			y_2		y_3		Correlations for Errors		
Parameter	$\beta_{1(y2)}$	$\beta_{1(y3)}$	$\beta_{1(X2)}$	$\beta_{2(X1)}$	$\beta_{2(X2)}$	$\beta_{3(X2)}$	$\beta_{3(X3)}$	r_{e12}	r_{e13}	r_{e23}
Values	**0.25**	**0.35**	**0.20**	**0.35**	**0.20**	**0.40**	**0.25**	**0.10**	**0.20**	**0.10**
SYSLIN (FIML)	0.2542 (0.0299)	0.3338 (0.0421)	0.2106 (0.0215)	0.3600 (0.0131)	0.1972 (0.0132)	0.4041 (0.0123)	0.2503 (0.0123)	NA	NA	NA
CALIS (ML)	0.2542 (0.0299)	0.3338 (0.0421)	0.2106 (0.0215)	0.3600 (0.0131)	0.1972 (0.0132)	0.4041 (0.0123)	0.2503 (0.0123)	0.0989	0.2135	0.1048
MIXED (ML)	0.2542 (0.0106)	0.3338 (0.0112)	0.2106 (0.0116)	0.3600 (0.0131)	0.1972 (0.0132)	0.4041 (0.0123)	0.2503 (0.0121)	0.0989	0.2135	0.1048
SYSLIN (SUR)	0.2980 (0.0106)	0.4803 (0.0112)	0.1324 (0.0114)	0.3604 (0.0131)	0.1971 (0.0132)	0.4040 (0.0123)	0.2511 (0.0123)	0.0146	0.0176	0.1047
SYSLIN (ITSUR)	0.2521 (0.0106)	0.3495 (0.0112)	0.2043 (0.0116)	0.3600 (0.0131)	0.1972 (0.0132)	0.4040 (0.0123)	0.2511 (0.0121)	0.0998	0.1943	0.1048

Note: Standard Errors are in parentheses under the parameter estimates.

Another approach to solving the parameter estimates is to set the equations up as a multivariate (or SUR in this case) linear mixed model (LMM) and use SAS PROC MIXED. However, multivariate LMMs have received scant treatment in the literature. Reinsel (1984) derived closed-form estimates with completely observed data and balanced designs. More recently, Shah, Laird, and Schoenfeld (1997) extended the EM-type algorithm of Laird and Ware (1982) to a bivariate (p = 2) setting. In econometric terminology, their model is analogous to SUR. Schafer and Yucel (2002) note that the added generality of the SUR model comes at a high cost, making the resulting algorithms impractical for more than a few response variables. Thus, it may be possible to recast the multivariate model as a univariate one by stacking the columns of y_j and applying SAS PROC MIXED with a user-specified covariance structure (see Appendix B for the code to stack the data). In most applications, however, this approach quickly becomes impractical. Examples for only p = 2 response variables with complete data (Shah et al., 1997) and incomplete data (Verbeke & Molenberghs, 2000) require complicated SAS macros. As the number of variables and number of individuals per cluster grows, the dimension of the response vector increases rapidly, and usage of SAS PROC MIXED can become practically impossible.

Fortunately, Park (1993) showed that the ML and IFGLS estimators are mathematically equivalent. As can be seen in Table 1 the estimates from PROC MIXED with an ML estimator and PROC CALIS with an ML estimator produce identical parameter estimates but slightly different standard errors. The results from PROC SYSLIN with the ITSUR option (IFGLS estimator) are virtually identical to those from PROC MIXED. PROC CALIS with an ML estimator and PROC SYSLIN with the FIML option produce identical parameter estimates and standard errors, but PROC SYSLIN does not report the correlation among the regression equations (error terms for the y variables). The SUR (FGLS) option gives similar results but the solution has not been iterated as in the ITSUR (IFGLS) option. A full-scale simulation study would be necessary to determine which approach would provide the most accurate and valid results. A researcher interested in conducting a simulation study could compare the bias in the coefficients and standard errors of the correctly specified regression (9) and SUR (10) models and the results from structural equation modeling software (e.g. SAS PROC CALIS). One could also assess power and Type I error of correctly specified and misspecified models. For example, on could analyze a model that incorrectly assumes a direct path from x_1 to y_1 and then investigate the Type I error rates produced by the different analytic approaches. Furthermore, one could compare the statistical properties of different estimation procedures under any of these circumstances.It would seem, however, that SAS PROC MIXED, although viable, may be inefficient due to computational demands.

Applications

There many situations in educational and behavioral research in which multiple dependent variables are of interest. Oftentimes these variables may take the pattern of path analytic model, but there are many other cases where they do not. However, it is commonplace for educational researchers to conduct separate analyses for multiple dependent variables even though they are likely to be correlated and have similar although not identical design matrices. For example, researchers in counseling often have multiple outcomes (measure of symptoms, coping, etc.) that are assumed to have some of the same predictors but to also have predictors that are unique to each measure. This is a situation that calls for a SUR model; however, a search of ERIC and PSYCHINFO located 11 applications of SUR models despite the enormous number of articles that analyze multiple dependent variables (see Appendix A). SUR models are underutilized and should be give more consideration as an analytic technique. The issue begins with education, and thus, we as statistics educators should devote more time to covering SUR models as a flexible analytic method in our multivariate analyses courses.

References

Laird, N. M. & Ware, J. H. (1982). Random-effects models for longitudinal data. *Biometrics*, *38*, 963-74.

Reinsel, G. (1984). Estimation and prediction in a multivariate random effects generalized linear model. *J. Amer. Statist. Assoc.*, *79*, 406-414.

Park, T. (1993). Equivalence of maximum likelihood estimation and iterative two-sage estimation for seemingly unrelated regression. *Communications in Statistics: Theory & Methods*, *22*, 2285-2296.

Schafer J. L, & Yucel, R. M. (2002). Computational Strategies for Multivariate Linear Mixed-Effects Models with Missing Values. *Journal of Computational and Graphical Statistics*, *11(2)*, 437-457.

Shah, A., Laird, N., Schoenfeld, D. (1997). A Random-Effects Model for Multiple Characteristics With Possibly Missing Data, *Journal of the American Statistical Association*, *92*, 775-779.

Timm, N. H. (2002). *Applied multivariate analysis*. New York: Springer.

Zellner, A. (1962). An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias. *Journal of the American Statistical Association*, *57*, 348-368.

Appendix A

References for Behavioral Science Articles using SUR

Godwin, D. D. (1985). Simultaneous Equations Methods in Family Research. *Journal of Marriage & the Family*, 47, 9-23.

Grofman, B., &; Migalski. (1988). Estimating the Extent of Racially Polarized Voting in Multicandidate Contests. *Sociological Methods & Research*, 16, 427-454.

Gruenewald, P. J. (1997). Analysis approaches to community evaluation. *Evaluation Review*, 21, 209-30.

Fernandez, S., Smith, C. R., & Wenger, J. B. (2007). Employment, Privatization, and Managerial Choice: Does Contracting out Reduce Public Sector Employment? *Journal of Policy Analysis & Management*, *26*, 57-77.

Hook, J. L. (2004). Reconsidering the Division of Household Labor: Incorporating Volunteer Work and Informal Support. *Journal of Marriage & the Family*, *66*, 101-118.

Lehrer, E. L. (1986). Simultaneous Equations Methods in Family Research: A Comment. *Journal of Marriage & the Family*, *48*, 881-883.

Minnotte, K. L., Stevens, D. P., Minnotte, M. C., & Kiger, G. (2007). Emotion-Work Performance among Dual-Earner Couples: Testing Four Theoretical Perspectives. *Journal of Family Issues*, *28*(*6*), 773-793.

Pelled,L. H., Eisenhardt, K. M., & Xin, K. R. (1999). Exploring the black box: An analysis of work group diversity, conflict, and performance. *Administrative Science Quarterly*, *44*(*1*), 1-28.

Price-Spratlen, T. (1998). Between depression and prosperity? Changes in the community context of historical African American migration. *Social Forces*, *77*, 515-539.

Schwartz, J. (2006). Effects of Diverse Forms of Family Structure on Female and Male Homicide. *Journal of Marriage & the Family*, *68*, 1291-1312.

Wu, Z. (2005). Generalized Linear Models in Family Studies. *Journal of Marriage & Family*, *67*, 1029-1047.

Zax, J. S. (2002). Comment on "Estimating the Extent of Racially Polarized Voting in Multicandidate Contests" by Bernard Grofman and Michael Migalski. *Sociological Methods & Research*, *31*, 75-87.

Send correspondence to: T. Mark Beasley
University of Alabama at Birmingham
Email: mbeasley@uab.edu

Appendix B
(SAS/IML Code to Generate Data for Figure 1)

```
proc iml;
N=5000;** sample Size **;
r_e12=0.10;r_e13=0.20;r_e23=0.10;** Correlation among Error terms **;
Rey=(1||r_e12||r_e13)//(r_e12||1||r_e23)//(r_e13||r_e23||1);
r_x12=0.30;r_x13=0.25;r_x23=0.15;** Correlation among Exogeneous Variables **;
Rxx=(1||r_x12||r_x13)//(r_x12||1||r_x23)//(r_x13||r_x23||1);
b1_y2=0.25;b1_y3=.35;********** Path Coefficients for Y1 **;
b1_x2=0.20;******************** Path Coefficients for Y1 **;
b2_x1=0.35;b2_x2=0.20;b2_x3=0;* Path Coefficients for Y2 **;
b3_x1=0;b3_x2=0.40;b3_x3=0.25;* Path Coefficients for Y3 **;
R2_y3=(b3_x1||b3_x2||b3_x3)*Rxx*((b3_x1||b3_x2||b3_x3)`);
R2_y2=(b2_x1||b2_x2||b2_x3)*Rxx*((b2_x1||b2_x2||b2_x3)`);
Rxxe=(Rxx||(j((nrow(Rxx)),1,0)))//(((j(1,(nrow(Rxx)),0)))||1);
vecre23=(0||0||0||r_e23);Rxxe=Rxxe//vecre23;Rxxe=Rxxe||((vecre23`)//1);
R_y23=(b2_x1||b2_x2||b2_x3||((1-R2_y2)##.5)||0)
*Rxxe*((b3_x1||b3_x2||b3_x3||0||((1-R2_y3)##.5))`);
R_y2x2=(b2_x1||b2_x2||b2_x3)*Rxx[,2];
R_y3x2=(b3_x1||b3_x2||b3_x3)*Rxx[,2];
Rxy1=(1||R_y23||R_y2x2)//(R_y23||1||R_y3x2)//(R_y2x2||R_y3x2||1);
print 'Rxx Correlation matrix for Y1' ;print Rxy1;
R2_y1=(b1_y2||b1_y3||b1_x2)*Rxy1*((b1_y2||b1_y3||b1_x2)`);
Rxy1e1=1||(((1-R2_y2)##.5)#r_e12)||(((1-R2_y3)##.5)#r_e13);
Rxy1e1=Rxy1e1//( (((1-R2_y2)##.5)#r_e12)||1||0);
Rxy1e1=Rxy1e1//( (((1-R2_y3)##.5)#r_e13)||0||1);
print 'Correlation Matirx for Y2-Y3-X2';print Rxy1e1;
R_y1e1=(b1_y2||b1_y3||b1_x2)*Rxy1*((b1_y2||b1_y3||b1_x2)`);
ry1e1=((((1-R2_y2)##.5)#r_e12)||(((1-R2_y3)##.5)#r_e13)||0)
*((b1_y2||b1_y3||b1_x2)`);ry1e1=ry1e1/(R2_y1##.5);
print 'Correlation of Y1-e1' ry1e1;
print 'R-squares';print R2_y1 R2_y2 R2_y3 R_y23 R_y2x2 R_y3x2;
seed=13;** Setting Seed gives the same Result everytime ;
*** For Errors of Y *******************;
lame=eigval(rey);*** LATENT ROOTS OF rey  *******************;
lsqrte=diag(lame##0.5);** DIAGONAL MATRIX WITH THE SQUARE ROOT OF EIGENVALUES;
eve=eigvec(rey);** EIGENVECTORS OF rey  ************************;
fre=eve*lsqrte;** CREATE FACTOR SCORE MATRIX (fre)      ******;
Ze= rannor(j(N,3,seed));Ze=fre*Ze`;Ze=Ze`;
*** For X variables *******************;
lamx=eigval(rxx);***  LATENT ROOTS OF Rxx  *******************;
lsqrtx=diag(lamx##0.5);** DIAGONAL MATRIX WITH THE SQUARE ROOT OF EIGENVALUES;
evx=eigvec(rxx);** EIGENVECTORS OF Rxx  ************************;
frx=evx*lsqrtx;** CREATE FACTOR SCORE MATRIX (frx)   ******;
Zx= rannor(j(N,3,0));Zx=frx*Zx`;Zx=Zx`;
***********************************;
e1=Ze[,1];e2=Ze[,2];e3=Ze[,3];
x1=Zx[,1];x2=Zx[,2];x3=Zx[,3];
y3=(b3_x1#x1)+(b3_x2#x2)+(b3_x3#x3)+(((1-R2_y3)##.5)#e3);
y2=(b2_x1#x1)+(b2_x2#x2)+(b2_x3#x3)+(((1-R2_y2)##.5)#e2);
qb=-2#(R2_y1##.5)#ry1e1;** Define the qb coefficient for Quadratic Equation *;
m=(qb+(((qb##2)-(4#(R2_y1-1)))##.5))/2;* Solve positive root of Quad Eq. ****;
print 'Coefficient for e1' m;
y1=(b1_y2#y2)+(b1_y3#y3)+(b1_x2#x2)+(m#e1);
dats=y1||y1a||y2||y3||x1||x2||x3||e1||e2||e3;
varname={'y1'  'y2' 'y3' 'x1' 'x2' 'x3' 'e1' 'e2' 'e3'};
create outs from dats [colname=varname];
   append from dats;
```

Appendix C
(SAS Code to Perform SUR Model and Path Analyses of Data from Figure 1)

```
data outs;set outs;id=_n_;run;
proc corr data=outs;run;
proc standard data = outs out=surpath mean=0 std=1;var y1 y2 y3 x1 x2 x3;run;
proc syslin data=surpath FIML; ** OTHER OPTIONS include SUR and ITSUR ***;
endogenous  y1 y2 y3;          **             INSTEAD of FIML        ***;
instruments x1 x2 x3;
      y1: model y1 = y2 y3    x2    / noint stb;
      y2: model y2 =       x1 x2    / noint stb;
      y3: model y3 =          x2 x3 / noint stb;run;
proc calis data=surpath method=ML;
LINEQS
      y3 = b3_x2 X2 + b3_x3 X3 + e_3,
      y2 = b2_x1 X1 + b2_x2 X2 + e_2,
      y1 = b1_y2 Y2 + b1_y3 Y3 + b1_x2 X2 + e_1;
STD X1=v_x1, X2=v_x2, X3=v_x3, e_3=v_e3, e_2=v_e2, e_1=v_e1;
COV e_1 e_2 = c_e12, e_1 e_3 = c_e13, e_2 e_3 = c_e23; run;
data stack;set surpath; ** STACKING THE DATA for PROC MIXED   ******;
do mod = 1 to 3;
  if mod = 1 then do;
   y=y1;b1_0=1;b1_y2=y2;b1_y3=y3;b1_x2=x2;
        b2_0=0;b2_x1= 0;b2_x2= 0;
        b3_0=0;b3_x2= 0;b3_x3= 0;
   output;
  end;
  if mod = 2 then do;
   y=y2;b1_0=0;b1_y2= 0;b1_y3= 0;b1_x2= 0;
        b2_0=1;b2_x1=x1;b2_x2=x2;
        b3_0=0;b3_x2= 0;b3_x3= 0;
   output;
  end;
  if mod = 3 then do;
   y=y3;b1_0=0;b1_y2= 0;b1_y3= 0;b1_x2= 0;
        b2_0=0;b2_x1= 0;b2_x2= 0;
        b3_0=1;b3_x2=x2;b3_x3=x3;
   output;
  end;
end;
run;
proc mixed data=stack method=ML   ;class mod id;
    model y =  b1_y2 b1_y3 b1_x2
               b2_x1 b2_x2
               b3_x2 b3_x3  /noint solution DDFM=KENWARDROGER   ;
    repeated mod / type=un subject=id r rcorr;run;
```